Pressure drop due to viscous flow through cylinders

By HOWARD BRENNER
Department of Chemical Engineering, New York University

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A general formula is developed which permits a calculation of the pressure drop arising from the slow steady flow of a viscous fluid through a circular cylinder for arbitrarily assigned conditions of velocity on the bounding surfaces of the cylinder. In particular, the diminution in pressure can be calculated *directly* from the prescribed boundary velocities without requiring a detailed solution of the equations of motion. Hence it is possible to compute, in comparatively simple fashion, the magnitude of this macroscopic parameter for a large variety of complex motions which would normally present great analytical difficulties.

By way of illustration the additional pressure drop arising from the presence of a point force situated along the axis of a cylinder is calculated. The additional force required to maintain the motion in the presence of the obstacle is exactly twice the magnitude of the point force itself.

One of the objectives of theoretical hydrodynamics is the calculation of various *macroscopic* parameters such as drag, lift, moment, pressure drop and the like from a knowledge of the stress field. In many instances one is less interested in the actual details of the fluid motion than in the numerical value of the parameters characterizing the flow. The adoption of this point of view is often imposed through purely mathematical difficulties occasioned by the complex structure of the fields. Accordingly, formulae which permit calculation of the macroscopic properties of the fluid motion *directly* from the boundary and initial conditions, without recourse to a detailed solution of the equations of motion, have much to commend them.

As a specific contribution in this general area we propose to develop an expression for the pressure drop associated with slow, steady viscous flows inside circular cylinders in terms of arbitrarily prescribed conditions of velocity over the bounding surfaces of the cylinder. This formula, coupled with a perturbation scheme known as the method of 'reflexions' (see, for instance, Brenner & Happel 1958), provides a useful tool for the study of the additional pressure drop caused by disturbances to a Poiseuille field of flow. These results, in turn, find application in the flow through dilute beds of particles such as in fluidized beds, pneumatic conveying, and similar processes (Happel & Brenner 1957).

Consider a time-independent viscous motion occurring within a circular cylinder of radius ρ_0 and length $2h$. It is presumed that the velocity and pressure fields $(\mathbf{v}, p)$ are well behaved at all points within the cylinder and that they there satisfy the creeping motion equation (in the absence of external forces)

$$\nabla^2 \mathbf{v} = \frac{1}{\mu} \nabla p \tag{1}$$

and the continuity equation

$$\nabla . \mathbf{v} = 0 \tag{2}$$

for incompressible fluids. In cylindrical co-ordinates (ρ, ϕ, z) the component velocities will be denoted by (v_ρ, v_ϕ, v_z).

The pressure drop in the positive z-direction, ΔP, is defined as the difference in pressure between the two planes, $z = -h$ and $z = h$, respectively, situated at either end of the cylinder; that is,

$$\Delta P = [p]_{z=-h} - [p]_{z=h}. \tag{3}$$

These planes are to be selected in such a way that the pressure across each is constant. Otherwise, our definition of pressure drop is ambiguous. In a majority of circumstances this requires that these planes be situated at $z = \pm\infty$.

In the absence of inertial and external forces the surface forces acting upon any fluid volume constitute a system of forces in equilibrium. Upon applying this argument to a cylinder of fluid of radius ρ bounded by the planes $z = \pm h$, and considering only the z-component of this force, we are led to the relation

$$\int_0^\rho \int_0^{2\pi} \{[P_{zz}]_{z=h} - [P_{zz}]_{z=-h}\} \rho \, d\phi \, d\rho + \int_{-h}^h \int_0^{2\pi} P_{\rho z} \rho \, d\phi \, dz = 0, \tag{4}$$

in which P_{zz} and $P_{\rho z}$ are the components of stress in the z-direction acting on the surfaces $z =$ constant and $\rho =$ constant, respectively.

For an incompressible viscous fluid these stresses can be written in the form

$$P_{zz} = -p + 2\mu \frac{\partial v_z}{\partial z}, \qquad P_{\rho z} = \mu\left(\frac{\partial v_z}{\partial \rho} + \frac{\partial v_\rho}{\partial z}\right).$$

However, according to the equation of continuity,

$$\frac{\partial v_z}{\partial z} = -\frac{\partial}{\rho \, \partial \rho}(\rho v_\rho) - \frac{\partial v_\phi}{\rho \, \partial \phi}$$

from which we eventually obtain

$$\int_0^\rho \int_0^{2\pi} \frac{\partial v_z}{\partial z} \rho \, d\phi \, d\rho = -\rho \int_0^{2\pi} v_\rho \, d\phi. \tag{5}$$

With the aid of these results and a simple partial integration, equation (4) can be put in the form

$$\Delta P \pi \rho^2 + \mu\rho \frac{\partial}{\partial \rho} \int_{-h}^h \int_0^{2\pi} v_z \, d\phi \, dz - \mu\rho \int_0^{2\pi} [v_\rho(\rho, \phi, h) - v_0(\rho, \phi, -h)] \, d\phi = 0.$$

If this relation is multiplied by $\rho \, d\rho$ and the resulting expression integrated from $\rho = 0$ to ρ_0 it takes the form

$$\frac{\Delta P \pi \rho_0^4}{4\mu} = -\rho_0^2 \int_{-h}^h \int_0^{2\pi} v_z(\rho_0, \phi, z) \, d\phi \, dz + 2 \int_{-h}^h \int_0^{\rho_0} \int_0^{2\pi} v_z \rho \, d\phi \, d\rho \, dz$$
$$+ \int_0^{\rho_0} \int_0^{2\pi} v_\rho(\rho, \phi, h) \rho^2 \, d\phi \, d\rho - \int_0^{\rho_0} \int_0^{2\pi} v_\rho(\rho, \phi, -h) \rho^2 \, d\phi \, d\rho.$$

Lastly, we can rid ourselves of the second integral in the above expression by setting $\rho = \rho_0$ in equation (5), multiplying both sides by dz and integrating the result from $z = -h$ to $z = z$. Upon doing this we finally obtain the desired formula,

$$\frac{\Delta P \pi \rho_0^4}{4\mu} = -\rho_0^2 \int_{-h}^{h} \int_0^{2\pi} v_z(\rho_0, \phi, z)\, d\phi\, dz + 4h \int_0^{\rho_0} \int_0^{2\pi} v_z(\rho, \phi, -h)\, \rho\, d\phi\, d\rho$$
$$-2\rho_0 \int_{z=-h}^{z=h} \int_{z'=-h}^{z'=z} \int_0^{2\pi} v_\rho(\rho_0, \phi, z')\, d\phi\, dz'\, dz + \int_0^{\rho_0} \int_0^{2\pi} v_\rho(\rho, \phi, h)\, \rho^2\, d\phi\, d\rho$$
$$-\int_0^{\rho_0} \int_0^{2\pi} v_\rho(\rho, \phi, -h)\, \rho^2\, d\phi\, d\rho. \tag{6}$$

This relation enables us to compute the pressure drop directly from the prescribed values of v_z on the walls and bottom of the container, $v_z(\rho_0, \phi, z)$ and $v_z(\rho, \phi, -h)$, respectively, and from the prescribed radial component on the cylinder walls, top and bottom, $v_\rho(\rho_0, \phi, z)$, $v_\rho(\rho, \phi, h)$ and $v_\rho(\rho, \phi, -h)$, respectively. In those applications where it is necessary to choose the constant pressure planes at $z = \pm\infty$ the velocity at infinity is zero and the second, fourth and fifth integrals in the foregoing expression vanish.

As a trivial illustration of the use of this formula we shall deduce Poiseuille's law for the diminution in pressure accompanying the laminar flow of a viscous fluid in a circular pipe. In this application we are, of course, making use of the fact that in the rectilinear flow of an incompressible fluid the creeping motion and Stokes–Navier equations are identical. The boundary conditions to be satisfied are:

(i) on the cylinder walls, $\rho = \rho_0$, $\quad v_z(\rho_0, \phi, z) = v_\rho(\rho_0, \phi, z) = 0$;

(ii) at the top of the cylinder, $z = h$, $\quad v_\rho(\rho, \phi, h) = 0$;

(iii) at the bottom of the cylinder, $z = -h$, $\quad v_\rho(\rho, \phi, -h) = 0$;

and
$$\int_0^{\rho_0} \int_0^{2\pi} v_z(\rho, \phi, -h)\, \rho\, d\phi\, d\rho = Q;$$

where Q is the volumetric flow rate through the cylinder and is independent of z. If $L = 2h$ is the length of pipe through which the fluid passes, we have from (6)

$$\frac{\Delta P \pi \rho_0^4}{4\mu} = 4(\tfrac{1}{2}L)\, Q, \qquad \text{or} \quad \Delta P = \frac{8\mu Q L}{\pi \rho_0^4}, \tag{7}$$

which is Poiseuille's law.

As a somewhat more instructive example, we consider the problem of an external force localized at a point ($\rho = 0, z = 0$), situated at the longitudinal axis of the cylinder. This force, whose magnitude is F, acts in the direction of the negative z axis. The pressure drop associated with this disturbance can be calculated from (6).

In the absence of the cylinder walls, when the fluid extends to infinity in all directions, the unperturbed motion, $(\mathbf{v}^{(0)}, p^{(0)})$, given by Lamb (1932) can be expressed in the form

$$\mathbf{v}^{(0)} = \frac{F}{4\pi\mu}\left[\tfrac{1}{2}\nabla\left(\frac{z}{r}\right) - \mathbf{i}_z \frac{1}{r}\right], \tag{8}$$

and
$$p^{(0)} = -\frac{F}{4\pi}\frac{z}{r^3}. \tag{9}$$

As a result of the presence of a boundary encircling the fluid, the original field of flow is perturbed by an amount $(\mathbf{v}^{(1)}, p^{(1)})$ and the entire motion $(\mathbf{v}, p)$ is

$$\mathbf{v} = \mathbf{v}^{(0)} + \mathbf{v}^{(1)}, \tag{10}$$

$$p = p^{(0)} + p^{(1)}. \tag{11}$$

The validity of this method of treatment depends upon the linearity of the equations of motion.

On the hypothesis of no slip at the cylinder walls we have the boundary condition

$$\mathbf{v} = 0 \quad \text{at} \quad \rho = \rho_0, \tag{12}$$

which is equivalent to

$$\mathbf{v}^{(1)} = -\mathbf{v}^{(0)} \quad \text{at} \quad \rho = \rho_0. \tag{13}$$

There are no further restrictions imposed on the field $\mathbf{v}^{(1)}$ except that it be free from singularities in the interior of the cylinder. As such, it cannot alter the force acting at the isolated point, $r = 0$, and the force is thus passed along unaltered to become a property of the field $\mathbf{v}$.

The planes across which the pressure, p, is constant occur at $z = \pm\infty$. On the basis of (11), the pressure drop due to the entire motion can be calculated from the relation

$$\Delta P = \Delta P^{(0)} + \Delta P^{(1)}. \tag{14}$$

But, equation (9) shows that the pressure drop due to the unperturbed motion is

$$\Delta P^{(0)} = [p^{(0)}]_{z=-\infty} - [p^{(0)}]_{z=\infty} = 0, \tag{15}$$

and hence

$$\Delta P = \Delta P^{(1)}. \tag{16}$$

Now $\Delta P^{(1)}$ can be computed from (6) by putting $h = \infty$. Since $\mathbf{v}^{(0)} \to 0$ as $z \to \pm\infty$ and since $\mathbf{v}^{(0)}$ and $\mathbf{v}^{(1)}$ are related by (13) then $v_z^{(1)}$ and $v_\rho^{(1)}$ vanish at the top and bottom of the cylinder, whence

$$\frac{\Delta P^{(1)} \pi \rho_0^4}{4\mu} = -\rho_0^2 \int_{-\infty}^{\infty} \int_0^{2\pi} v_z^{(1)}(\rho_0, z)\, d\phi\, dz - 2\rho_0 \int_{z=-\infty}^{z=\infty} \int_{z'=-\infty}^{z'=z} \int_0^{2\pi} v_\rho^{(1)}(\rho_0, z')\, d\phi\, dz'\, dz. \tag{17}$$

In view of equations (13) and (14) and the lack of dependence of the velocity field on the angle ϕ, the above equation becomes

$$\frac{\Delta P \pi \rho_0^4}{4\mu} = 2\pi\rho_0^2 \int_{-\infty}^{\infty} v_z^{(0)}(\rho_0, z)\, dz + 4\pi\rho_0 \int_{z=-\infty}^{z=\infty} \int_{z'=-\infty}^{z'=z} v_\rho^{(0)}(\rho_0, z')\, dz'\, dz. \tag{18}$$

From (8)

$$v_z^{(0)}(\rho_0, z) = \frac{F}{4\pi\mu}\left[\frac{1}{2}\frac{\partial}{\partial z}\left(\frac{z}{\sqrt{(\rho_0^2 + z^2)}}\right) - \frac{1}{\sqrt{(\rho_0^2 + z^2)}}\right],$$

and

$$v_\rho^{(0)}(\rho_0, z') = \frac{F}{8\pi\mu} z' \left[\frac{\partial}{\partial \rho}\left(\frac{1}{\sqrt{(\rho^2 + (z')^2)}}\right)\right]_{\rho=\rho_0}$$

$$= \frac{F}{8\pi\mu}\rho_0 \frac{\partial}{\partial z'}\left(\frac{1}{\sqrt{(\rho_0^2 + (z')^2)}}\right),$$

which easily yields

$$\int_{z'=-\infty}^{z'=z} v_\rho^{(0)}(\rho_0, z')\,dz' = \frac{F}{8\pi\mu}\frac{\rho_0}{\sqrt{(\rho_0^2+z^2)}}.$$

Inserting these values in (18) it becomes

$$\frac{\Delta P\pi\rho_0^4}{4\mu} = \frac{F\rho_0^2}{4\mu}\int_{-\infty}^{\infty}\frac{\partial}{\partial z}\left(\frac{z}{\sqrt{(\rho_0^2+z^2)}}\right)dz, \tag{19}$$

from which we find $\Delta P = 2F/\pi\rho_0^2$. The pressure drop force, $\Delta P\pi\rho_0^2$, is thus exactly twice the external force which gives rise to it.

If the external force arises from the action of a viscous fluid impinging on a small spherical particle situated at the cylinder axis, then F is equal to and oppositely directed from the drag, D, experienced by the particle while ΔP is the *additional** pressure drop caused by its presence. Without further calculation, Faxén's (1927) law shows that for a small particle

$$D = -F = -6\pi\mu a(U_0 - U), \tag{20}$$

where $\frac{1}{2}U_0$ is the superficial velocity of the fluid, U is the velocity of the particle in the z-direction and a is the particle radius. Under these circumstances

$$\Delta P = \frac{12\mu a(U_0 - U)}{\rho_0^2}. \tag{21}$$

For small values of a/ρ_0, this result agrees identically with that of Happel & Byrne (1954) obtained by a much more complex procedure requiring a detailed calculation of the field $\mathbf{v}^{(1)}$.

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* The fluid experiences a pressure drop even in the absence of the obstacle. The difference between the total pressure drop in the presence of the particle and the Poiseuillian pressure drop in the absence of the particle is the *additional* pressure drop. It is an experimentally measurable quantity.